Excitation gaps of incompressible composite fermion states: Approach to the Fermi sea

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Activation gaps are determined for fractional quantum Hall states with up to seven filled Landau levels of composite fermions carrying two vortices, which correspond to electron filling factors of up to 7/15 along the sequence $\nu=n/(2n+1)$. Systems with as many as 100 composite fermions are studied for this purpose. The relevance of the results to the issues of composite-fermion mass and the stability of fractional quantum Hall effect along the sequence $n/(2n+1)$ is discussed.

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I. INTRODUCTION

The mass of the composite fermion[1–3] (CF) has been defined[4,5] by interpreting the excitation gap as the effective cyclotron energy of composite fermions.[6] At the filling factors $\nu=n/(2pn\pm1)$, which correspond to n filled composite-fermion Landau levels (LL's), the gap is expressed as

$$\Delta_{n/(2pn\pm1)}\equiv\frac{\hbar eB^*}{m^*c}=\frac{1}{(2pn\pm1)}\frac{\hbar eB}{m^*c}, \tag{1}$$

where B is the external magnetic field, $B^*=B/(2pn\pm1)$ is the effective magnetic field, m^* is the composite-fermion mass, and $2p$ is the vorticity of the composite fermion. From dimensional considerations, we also know that the gap is proportional to the Coulomb interaction energy, for which convenient units are $e^2/\epsilon l_0$, where ϵ is the dielectric constant of the background semiconductor, $l_0=\sqrt{\hbar c/eB}$ is the magnetic length. This implies that, for a given filling factor, m^* depends on the magnetic field as $m^*\sim\sqrt{B}$, which is dictated by the fact that the mass is generated by the interaction. In order for the mass to be filling factor independent (apart from the $\sqrt{B}$ dependence) for composite fermions of a given vorticity $(2p)$, the gap must satisfy

$$\Delta_{n/(2pn\pm1)}=\frac{C_p}{(2pn\pm1)}\frac{e^2}{\epsilon l_0}. \tag{2}$$

An n-independent C_p would imply a filling factor independent mass $m^*=(\hbar^2\epsilon)/(C_pe^2l_0)$. (Of course the mass would depend on the CF vorticity $2p$.) Our aim in this paper will be to test Eq. (2), appearing very naturally within the composite-fermion framework, for more fractions than has been done in the past. Given that the composite-fermion mass is generated entirely from the Coulomb interaction, it would not be surprising, and certainly not inconsistent with any fundamental principle, if it depended in some way on the filling factor. However, if the dependence on the filling factor turns out to be too severe, the concept of the composite-fermion mass would cease to be useful. Therefore, it is of interest to investigate the filling factor dependence of the gap along the sequence of fractions for a given $2p$. Another motivation is to see if a logarithmic divergence predicted in Ref. 4 can be detected. Of course, the gap is also of fundamental interest in its own right, being directly responsible for the fractional quantum Hall effect.

In experiments, the composite-fermion mass has been measured by several methods.[5,7–14] For the sequence $\nu=n/(2n+1)$, all of the methods find the CF mass to be much larger than the electron band mass in GaAs ($0.07m_e$, where m_e is the electron mass in vacuum), and roughly comparable to m_e for typical experimental parameters. The analysis of the gaps measured from transport data is somewhat complicated by the presence of disorder, which appears to have a significant effect, especially when the gaps are small; nonetheless, a phenomenological treatment of disorder[5] in terms of a constant composite-fermion level broadening provides a consistent picture in terms of a filling factor independent mass. The mass deduced from the temperature dependence of the Shubnikov–de Haas oscillations[9] shows an enhancement at 7/15, suggesting that it may have substantial filling factor dependence for large n.

This question also pertains to the issue of the stability of the fractional quantum Hall effect (FQHE) along a sequence $n/(2pn\pm1)$. In the ideal situation, will FQHE occur for arbitrarily large n, or will the sequence terminate at some point? (We note here that many FQHE states that can occur *in principle* do not occur in reality for the Coulomb interaction; for example, at very small fillings or in higher Landau levels, the FQHE states are unstable for the Coulomb interaction.[15,16]) While it is not guaranteed by any means that our calculations will have the necessary accuracy to capture such an effect, they might reveal trends indicating such a termination. In Ref. 15, it was found that the gap at 5/11 is below the prediction of Eq. (2); is that an indication of a real trend or merely a fluctuation? Experimentally, there exists evidence[17] for at least ten members of the sequence $n/(2n+1)$.

Motivated by these considerations, we have undertaken a calculation of the gaps along the sequence $\nu=n/(2n+1)$ and will report results up to $\nu=7/15$. The calculation uses wave functions for composite fermions for the ground and the excited states. The ground state has n filled levels of composite fermions and the excited state contains a far separated particle-hole pair of composite fermions. The wave functions have been tested for 1/3, 2/5, and 3/7 against exact diagonalization results, and the gaps predicted from these wave functions are accurate to within 2–3 %.[18,15] No such independent tests exist for other fractions, but there is no

reason to expect the wave functions to suddenly deteriorate; in this context, it may be worth mentioning that the accuracy of gaps at 1/3, 2/5, and 3/7 from these wave functions is quite comparable. Nonetheless, the results below are only as accurate as the wave functions.

The gaps for filling factors $\leqslant 5/11$ were reported in Ref. 15 using this method, with an investigation of up to 40 particles. The system sizes were too small for accurate estimates, especially for 5/11, because the largest wave vector achieved in the study was still in the range that is affected by the CF particle-hole interaction.[19] In the present work, we use parallel computers to push the Monte Carlo calculation up to 7/15 using systems with as many as 100 composite fermions. We get more accurate numbers for the gaps reported earlier, as well as gaps for new FQHE states.

The gap, an $O(1)$ energy, is obtained as the difference between the $O(N)$ energies of the ground and excited states. The calculation becomes increasingly more time consuming as one goes along $n/(2n+1)$ for two reasons. First, a study of large systems is required in order to investigate higher-order states; for example, 7/15 requires a minimum of 49 electrons in the spherical geometry. Second, the energies of the ground and excited states need to be determined with greater accuracy both because the energies increase with N and the energy difference decreases with n. For 100 composite fermions, the energies at 7/15 must be determined to an accuracy of better than 0.02% in order to get a reasonable value for the gap. A single gap value for 100 particles requires on the order of 10^8 total Monte Carlo steps, taking $\sim 20\,000$ CPU hours on a fast workstation. The results reported below have taken altogether $\approx 10^9$ Monte Carlo iterations. We divide all of the Monte Carlo steps into several configurations. The error is given by the standard deviation in energies over L configurations

$$\sigma = \sqrt{\frac{1}{L(L-1)} \sum_{i=1}^{L} \sum_{j=1}^{L} \langle (E_i - \langle E \rangle)(E_j - \langle E \rangle) \rangle}, \quad (3)$$

where the angular brackets denote an average over Monte Carlo steps and E_i is the energy of the ith configuration. We place each configuration on a single node of a Beowolf class PC cluster. One node consists of a dual 1 GHZ Pentium III processor. To obtain one data point, at a particular N, we use as many as 30 nodes repeatedly until the standard deviation in energies is sufficiently low to produce the desired accuracy.

II. CALCULATIONAL METHOD

The wave functions at $\nu = n/(2pn+1)$ are given by[1]

$$\Psi_\nu = P_{LLL} \Phi_1^{2p} \Phi, \quad (4)$$

where Φ_1 is the Slater determinant of single-particle eigenstates filling the lowest Landau level, Φ is an antisymmetric wave function for free fermions at an effective filling n, and P_{LLL} is the lowest Landau-level projection operator.

We work in the spherical geometry in which N electrons move on the surface of a sphere under the influence of a radial magnetic field B created by a magnetic monopole at the center.[20,21] The magnitude of the B field is given by $2Q\phi_0/4\pi R^2$ where $\phi_0 = hc/e$ is known as the flux quantum, R is the radius of the sphere, and Q is called the monopole strength which should be either an integer or a half-integer because of Dirac's quantization condition.

The interacting electron system at monopole strength Q maps into a system of weakly interacting composite fermions at an effective monopole strength $q = Q - p(N-1)$. In order to write the wave functions for composite-fermion ground and excited states at q, we first consider the wave functions for noninteracting electrons at q, denoted by Φ. Φ is in general a linear superposition of Slater-determinant basis states made up of the monopole harmonics,[21] $Y_{q,\bar{n},m}$, given by

$$\begin{aligned} Y_{q,\bar{n},m}(\Omega_j) = {} & N_{q\bar{n}m}(-1)^{q+\bar{n}-m} e^{iq\phi_j} u_j^{q+m} v_j^{q-m} \\ & \times \sum_{s=0}^{\bar{n}} (-1)^s \binom{\bar{n}}{s} \binom{2q+\bar{n}}{q+\bar{n}-m-s} \\ & \times (v_j^* v_j)^{\bar{n}-s} (u_j^* u_j)^s, \end{aligned} \quad (5)$$

where

$$N_{q\bar{n}m} = \left(\frac{(2q+2\bar{n}+1)}{4\pi} \frac{(q+\bar{n}-m)!(q+\bar{n}+m)!}{\bar{n}!(2q+\bar{n})!} \right)^{1/2}. \quad (6)$$

$\bar{n} = 0,1,2,\ldots$, is the LL index (to be differentiated from n, the number of filled Landau levels). Ω_j represents the angular coordinates θ_j and ϕ_j of the jth electron, and

$$u_j \equiv \cos(\theta_j/2)\exp(-i\phi_j/2), \quad (7)$$

$$v_j \equiv \sin(\theta_j/2)\exp(i\phi_j/2). \quad (8)$$

The free electron wave functions for the ground and excited states with n filled LL's on the sphere are then

$$\Phi_n^{gd} = \mathrm{Det}[Y_{q,\bar{n},m}]$$

$$\Phi_n^{ex} = c^+_{\bar{n}_t+1,-(q+\bar{n}_t)} c^-_{\bar{n}_t,(q+\bar{n}_t)} \mathrm{Det}[Y_{q,\bar{n},m}] \quad (9)$$

where $c^-_{\bar{n},m}$, ($c^+_{\bar{n},m}$) annihilates, (creates) and electron in the $\bar{n}$th LL. Here $\bar{n}_t$ is the LL index of the topmost filled LL. Φ_n^{ex} places one particle in the $\bar{n}_t+1$ LL, at the north pole of the sphere, and a hole in the $\bar{n}_t$ LL, at the south pole.

It was shown in Refs. 15 that the corresponding wave function of composite fermions at q, which gives the wave function of interacting electrons at $Q = q + p(N-1)$, is obtained from Φ by replacing $Y_{q,\bar{n},m}$ by $Y^{CF}_{q,\bar{n},m}$, defined as

$$Y^{CF}_{q,n,m}(\Omega_j) = N_{q\bar{n}m}(-1)^{q+\bar{n}-m}\frac{(2Q+1)!}{(2Q+\bar{n}+1)!}u_j^{q+m}v_j^{q-m} \times \sum_{s=0}^{\bar{n}}(-1)^s\binom{\bar{n}}{s}\binom{2q+\bar{n}}{q+\bar{n}-m-s}u_j^s v_j^{\bar{n}-s} \times\left[\left(\frac{\partial}{\partial u_j}\right)^s\left(\frac{\partial}{\partial v_j}\right)^{\bar{n}-s}J_j^p\right], \quad (10)$$

where

$$J_j = \prod_k{}' (u_j v_k - v_j u_k). \quad (11)$$

Here the prime denotes the condition $k \neq j$. The ground- and excited-state CF wave functions are obtained from the corresponding integral quantum Hall wave functions by the replacement $Y \to Y^{CF}$.

We evaluate the energies of these wave functions for the Coulomb interaction by Monte Carlo. In order to deal with the derivatives we find it convenient to write them as follows:

$$\left(\frac{\partial}{\partial u_j}\right)^s\left(\frac{\partial}{\partial v_j}\right)^{\bar{n}-s}J_j^p = J_j^p[\bar{U}_j^s\bar{V}_j^{\bar{n}-s}1], \quad (12)$$

where

$$\bar{U}_j = J_j^{-p}\frac{\partial}{\partial u_j}J_j^p = p\sum_k{}'\frac{v_k}{u_j v_k - v_j u_k} + \frac{\partial}{\partial u_j},$$

$$\bar{V}_j = J_j^{-p}\frac{\partial}{\partial v_j}J_j^p = p\sum_k{}'\frac{-u_k}{u_j v_k - v_j u_k} + \frac{\partial}{\partial v_j}. \quad (13)$$

For a given $\bar{n}$, the explicit analytical form of the derivatives is used in the evaluation of the wave function.

Y^{CF} thus require evaluation of

$$P_j(s,\bar{n}-s) \equiv [\bar{U}_j^s\bar{V}_j^{\bar{n}-s}1], \quad (14)$$

where all $P_j(s,\bar{n}-s)$ with $s=0,\ldots,\bar{n}$ will be needed for the nth CF level. The explicit form for $P_j(s,\bar{n}-s)$ is given in the Appendix for up to the 6th CF level.

In this paper, we will be concerned with the sequence $\nu = n/(2n+1)$, i.e., with $p=1$. The ground state here is given by n filled levels of composite fermions. A particle-hole pair of composite fermions is obtained by exciting a composite fermion from one of the filled CF-LL's to an empty CF-LL; the lowest energy is obtained when the excitation takes place from the topmost occupied CF-LL into the lowest unoccupied CF-LL. The gap measured in transport experiments is the energy required to create a far separated CF particle-hole pair. In the spherical geometry, the farthest separation is obtained by putting the particle and the hole at the north and south poles. With the explicit wave functions, we obtain the energies of the ground and excited states by Monte Carlo for the Coulomb interaction. The gap so obtained also includes the interaction energy of the CF particle and hole. Even though this energy vanishes when the two are far away, we explicitly subtract from our gaps $-e^2/[(2n+1)^2 2R]$, the interaction energy of a particle and hole of charge $|e^*| = e/(2n+1)$ at the opposite poles, before we extrapolate the results to the $1/N \to 0$ limit. We have not studied corrections to the mass due to finite thickness, disorder, or Landau-level mixing here, which are presumably substantial for the states with a large number of filled composite-fermion levels. Our goal here is not to compare with experiment, but to ask how well the concept of mass works in the ideal situation when these effects are absent.

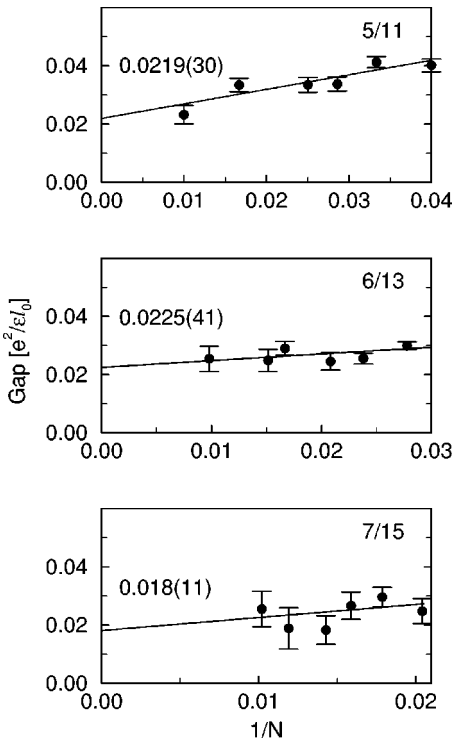

FIG. 1. Extrapolation of the activation gaps to the thermodynamic limit for $\nu = 5/11$, 6/13, and 7/15. N is the number of composite fermions. The error bars denote the Monte Carlo uncertainty and the solid line is a χ^2 straight-line fit.

III. RESULTS AND DISCUSSION

Figure 1 shows the gaps at 5/11, 6/13, and 7/15 as function of N^{-1}, N being the number of composite fermions. The thermodynamic limit obtained from a linear χ^2 fit is plotted in Fig. 2 against $(2n+1)^{-1}$, also including the gaps for the other FQHE states, taken from the literature (see Ref. 15 and references therein).

We note that for even with 100 particles, there are substantial finite-size oscillations for higher-order fractions such as 7/15, presumably due to the distance dependence of the CF particle-hole interaction, indicating that the distance between them is not large compared to their size even for 100 particles.[24] That, combined with the statistical uncertainty in Monte Carlo, makes an accurate estimation of the gap difficult, leading to fairly large error bars, especially at 7/15. Nonetheless, the gaps remain positive, providing insight into

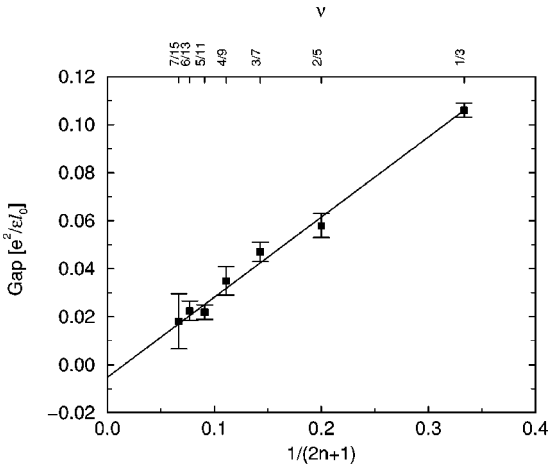

FIG. 2. Thermodynamic values of the gaps plotted as a function of $1/(2n+1)$. The best straight-line fit is given by $0.333(16)(2n+1)^{-1}-0.0052(30)$.

the remarkable stability[17] of FQHE along $n/(2n+1)$.

The plot of gaps as a function of $1/(2n+1)$, shown in Fig. 2, is consistent with the straight-line behavior of Eq. (2). However, due to the large error bars, it is not possible to definitively rule out other functional dependences, in particular, a logarithmic correction to the straight line. If one fits the gap with a straight line, the best fit is given by

$$\Delta_{n/(2n\pm1)}=\left[\frac{0.333(16)}{(2n+1)}-0.0052(30)\right]\frac{e^2}{\epsilon l_0}, \tag{15}$$

which gives an activation mass of $m^*\approx 3(\hbar^2\epsilon/l_0e^2)$. We note that the intercept at $\nu=1/2$ is slightly negative, which might imply an instability at a large n; however, given the uncertainty in the functional form of the fitting curve, as well as large error bars, no definitive conclusions may be drawn.

In the end, we note that a recent work[22] has estimated gaps for 1/3, 2/5, 3/7, and 4/9 by exact diagonalization on systems with up to 18 particles. The gaps reported there are 10–20 % lower than those calculated here and deviate from Eq. (2). Further work will be required to determine whether the differences arise from finite-size effects, different extrapolation, schemes, or imply logarithmic corrections to the wave functions Ψ_ν.

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APPENDIX

For the projected wave functions, we need an explicit expression for

$$P_j(s,\bar{n}-s)\equiv[\bar{U}_j^s\bar{V}_j^{\bar{n}-s}1] \tag{A1}$$

with

$$\bar{U}_j=J_j^{-p}\frac{\partial}{\partial u_j}J_j^p=p{\sum_k}'\frac{v_k}{u_jv_k-v_ju_k}+\frac{\partial}{\partial u_j},$$

$$\bar{V}_j=J_j^{-p}\frac{\partial}{\partial v_j}J_j^p=p{\sum_k}'\frac{-u_k}{u_jv_k-v_ju_k}+\frac{\partial}{\partial v_j}, \tag{A2}$$

which we evaluate by the following method.[23]

We first define

$$f_j(\alpha,\beta)\equiv{\sum_k}'\left(\frac{v_k}{u_jv_k-v_ju_k}\right)^\alpha\left(\frac{-u_k}{u_jv_k-v_ju_k}\right)^\beta. \tag{A3}$$

Equations (A2) can be rewritten as

$$\bar{U}_j=pf_j(1,0)+\frac{\partial}{\partial u_j}, \tag{A4}$$

$$\bar{V}_j=pf_j(0,1)+\frac{\partial}{\partial v_j}. \tag{A5}$$

Also, the derivatives of $f_j(\alpha,\beta)$ with respect to u_j and v_j have the simple form

$$\frac{\partial}{\partial u_j}f_j(\alpha,\beta)=-(\alpha+\beta)f_j(\alpha+1,\beta),$$

$$\frac{\partial}{\partial v_j}f_j(\alpha,\beta)=-(\alpha+\beta)f_j(\alpha,\beta+1). \tag{A6}$$

Explicit expressions for $P_j(s,\bar{n}-s)$ in terms of $f(\alpha,\beta)$ for various values of arguments are given below for the lowest six CF-Landau levels.[23] (For yet higher Landau levels, the expressions become too long to reproduce here, but can be evaluated following the same method using MATHEMATICA.) $P_j(\bar{n}-s,s)$ can be obtained from $P_j(s,\bar{n}-s)$ by swapping the arguments of all $f_j(\alpha,\beta)$ on the right-hand side. Therefore we will list only $P_j(s,\bar{n}-s)$ with $s\geqslant\bar{n}-s$. In the following, the subscript j on the quantities P and f will be left as implicit.

For $\bar{n}=0$,

$$P(0,0)=1.$$

For $\bar{n}=1$,

$$P(1,0)=pf(1,0).$$

For $\bar{n}=2$,

$$P(2,0)=p^2f(1,0)^2-pf(2,0),$$

$$P(1,1)=p^2f(0,1)f(1,0)-pf(1,1).$$

For $\bar{n}=3$,

$$P(3,0)=p^3\ f(1,0)^3-3\ p^2\ f(1,0)f(2,0)+2\ pf(3,0),$$

$$P(2,1)=p^3f(0,1)f(1,0)^2-2p^2f(1,0)f(1,1)-p^2f(0,1)f(2,0)+2pf(2,1).$$

For $\bar{n}=4$,

$$P(4,0)=p^4\,f(1,0)^4-6\,p^3\,f(1,0)^2\,f(2,0)+3\,p^2\,f(2,0)^2+8\,p^2\,f(1,0)f(3,0)-6\,pf(4,0),$$

$$P(3,1)=p^4f(0,1)f(1,0)^3-3p^3f(1,0)^2f(1,1)-3p^3f(0,1)f(1,0)f(2,0)+3p^2f(1,1)f(2,0)+6p^2f(1,0)f(2,1)+2p^2f(0,1)f(3,0)-6pf(3,1),$$

$$P(2,2)=p^4f(0,1)^2f(1,0)^2-p^3f(0,2)f(1,0)^2-4p^3f(0,1)f(1,0)f(1,1)+2p^2f(1,1)^2+4p^2f(1,0)f(1,2)-p^3f(0,1)^2f(2,0)+p^2f(0,2)f(2,0)+4p^2f(0,1)f(2,1)-6pf(2,2).$$

For $\bar{n}=5$,

$$P(5,0)=p^5f(1,0)^5-10p^4f(1,0)^3f(2,0)+15p^3f(1,0)f(2,0)^2+20p^3f(1,0)^2f(3,0)-20p^2f(2,0)f(3,0)-30p^2f(1,0)f(4,0)+24\,pf(5,0),$$

$$P(4,1)=p^5f(0,1)f(1,0)^4-4p^4f(1,0)^3f(1,1)-6p^4f(0,1)f(1,0)^2f(2,0)+12p^3f(1,0)f(1,1)f(2,0)+3p^3f(0,1)f(2,0)^2+12p^3f(1,0)^2f(2,1)-12p^2f(2,0)f(2,1)+8p^3f(0,1)f(1,0)f(3,0)-8p^2f(1,1)f(3,0)-24p^2f(1,0)f(3,1)-6p^2f(0,1)f(4,0)+24pf(4,1),$$

$$P(3,2)=p^5f(0,1)^2f(1,0)^3-p^4f(0,2)f(1,0)^3-6p^4f(0,1)f(1,0)^2f(1,1)+6p^3f(1,0)f(1,1)^2+6p^3f(1,0)^2f(1,2)-3p^4f(0,1)^2f(1,0)f(2,0)+3p^3f(0,2)f(1,0)f(2,0)+6p^3f(0,1)f(1,1)f(2,0)-6p^2f(1,2)f(2,0)+12p^3f(0,1)f(1,0)f(2,1)-12p^2f(1,1)f(2,1)-18p^2f(1,0)f(2,2)+2p^3f(0,1)^2f(3,0)-2p^2f(0,2)f(3,0)-12p^2f(0,1)f(3,1)+24pf(3,2).$$

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[24] The oscillations in the CF exciton dispersion (Ref. 19) are a consequence of the interaction between the CF particle and the CF hole. These lead to oscillations in the gap as a function of N, because the wave vector at the largest particle-hole separation increases with N. For sufficiently large wave vectors (recall that the particle-hole separation is proportional to the wave vector),

the dispersion is expected to be independent of the wave vector, but, from Ref. 19, we know that even for $\nu=2/5$ and 3/7 this does not happen up to $kl_0 \approx 2$, where k is the wave vector. For 7/15, we expect that the effects due to particle-hole interaction will persist up to still larger kl_0, because the particle and the hole are more spread out because of the larger effective magnetic length. For 98 particles at $\nu=7/15$, the wave vector of the particle-hole pair at maximum separation is $kl_0=L_{max}/\sqrt{Q} \approx 2$, which suggests that finite-size effects are not negligible even for 98 particles at this filling factor.