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Cold-atom optical lattices as quantum analog simulators for aperiodic one-dimensional localization without disorder

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Cold-atom optical lattices allow for the study of quantum localization and mobility edges in a disorder-free environment. We predict the existence of an Anderson-like insulator with sharp mobility edges in a one-dimensional nearly periodic optical lattice. We show that the mobility edge manifests itself as the early onset of pinning in center of mass dipole oscillations in the presence of a magnetic trap, which should be observable in optical lattices.
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Optical lattices incorporating ultracold-atomic condensates are rapidly becoming ideal quantum systems for studying various model Hamiltonians developed earlier for studying solid-state phenomena. This is primarily due to the extraordinary level of precise tunability that experimentalists have achieved in controlling the parameters (e.g., hopping, interaction, and lattice periodicity) of the optical lattice, which makes it possible for the cold-atom optical lattice to operate as an ideal quantum analog simulator for various many-body condensed-matter Hamiltonians. By contrast, ideal model Hamiltonians (e.g., Hubbard and Anderson models) often poorly describe solid-state systems, since experimental control over complex condensed-matter systems is, in general, quite limited. In addition, solid-state systems are invariably contaminated by unknown disorder, defects, and impurities whose effects are not easy to incorporate in model Hamiltonians. The cold-atom optical lattices are, therefore, becoming increasingly important in advancing our knowledge about the quantum phase diagram and the crossover in model many-body Hamiltonians of intrinsic interest. Examples include: the Bose-Hubbard model [1], the Tonks-Girardeau gas [2], and the Bose-Einstein-Condensation–Bardeen-Cooper-Schrieffer (BEC-BCS) crossover [3].

In addition to studying strong correlation effects (e.g., the superfluid-Mott insulator transition in the Bose-Hubbard model) in many-body Hamiltonians, cold-atom optical lattices also offer ideal systems for studying quantum transport phenomena including ballistic quantum transport [4–6] and quantum localization [7–10]. The latter may be more generally classified as metal-insulator transition phenomena with a direct relationship to the solid state. The distinction between a "metal" (i.e., a system with finite resistivity at zero temperature) and an "insulator" (i.e., a system with infinite zero temperature resistivity) is purely quantum. Broadly speaking, there are four classes of metal-insulator transitions in quantum lattice systems: Metal-band-insulator transition in an ordered periodic lattice arising from the chemical potential moving into energy band gaps; interaction induced metal-insulator transition as in the Mott transition; disorder-induced quantum localization (i.e., Anderson localization [11]); and quantum localization in aperiodic (but deterministic) potentials in disorder-free lattice systems.

In this paper, we establish that very general aspects of the metal-insulator transition phenomena (in the disorder-free environment) can be directly experimentally studied in aperiodic cold-atom optical lattices with the tuning of the experimental parameters leading to the observation of *both* band and quantum (Anderson-like) localization in the same system but in different parameter regimes. Such an experimental study of localization or insulating transitions in deterministic aperiodic systems is impossible in solid-state lattice systems, since disorder (which leads to direct Anderson localization) is invariably present in solid-state systems overwhelming any subtle localization effects arising from deterministic aperiodic potentials. In particular, all states are localized in one-dimensional systems in the presence of any disorder, whereas one-dimensional aperiodic potentials allow for the existence of extended quantum eigenstates. This makes one-dimensional optical lattice systems particularly interesting from the perspective of localization studies in deterministic aperiodic potentials, since such studies in the corresponding one-dimensional solid-state systems are essentially impossible due to disorder effects. We, therefore, consider aperiodic quantum localization in one-dimensional optical lattices, conclusively establishing the feasibility of studying this unusual phenomenon in cold-atom optical lattices.

The single-particle quantum localization problem in a deterministic quasiperiodic potential (i.e., two lattice potentials with mutually incommensurate periods) has a long history [12,13]. In particular, localization properties have been extensively studied in the Harper (or, equivalently, Aubry) model, which has an intriguing self-dual point where the eigenstates form a multifractal Cantor set spectra, and are neither localized nor extended. Away from the dual point, conventional wisdom dictates that all states, as a function of the chemical potential, are either all extended or all localized, depending on the mutual strengths of the potential and hopping terms. Such Harper-model-type quasiperiodic potentials, therefore, do not allow for the existence of a mobility edge separating the extended states (above the mobility edge) from the localized states (below the mobility edge), which is the hallmark of the Anderson localization transition in three-dimensional disordered systems. Central to our work is the conclusive theoretical demonstration of a class of one-dimensional optical lattice systems where the deterministic lattice potential *does* allow for the existence of a mobility edge in one dimension [14], which cannot happen through Anderson localization with disorder. This class of models distinguishes itself from other models discussed in the con-

text of optical lattices [15] through the formation of a metal-insulator mobility edge rather than a metal-band edge. We find that: (1) Direct numerical simulation and an analytic WKB approximation provide conclusive evidence for a rare metal-insulator mobility edge in a one-dimensional model, the nearly periodic Harper model. (2) Transport measurements in suitably designed, one-dimensional optical lattices can exhibit the mobility edge.

We consider spinless fermions (or equivalently hardcore bosons sufficiently near the Tonks-Girardeau regime) in the lowest band of a one-dimensional, tight-binding lattice with external potentials:

$$-u_{n+1}-u_{n-1}+(V_n+V_D f_n+\Omega n^2)u_n=Eu_n, \qquad (1)$$

where the amplitudes u_n multiply the Wannier states at sites n in the real-space wave function $\Psi(x)=\Sigma_n u_n w(x-n)$. We work in units of the hopping matrix element, $t=1$, and lattice spacing of the primary lattice defining the tight-binding problem, $a=1$, unless otherwise noted. The statistics of spinless fermions implicitly allow for an arbitrary on-site interaction in the above single-band model. In the absence of an external potential, the solutions form extended states, $u_n=u_0\exp(in\phi)$, with band energies $E=2\cos(\phi)$, for $0\leqslant\phi\leqslant\pi$. The band edges lie at $E=\pm 2$. In the presence of an oscillatory modulation of strength V, much weaker than the primary lattice, we can ignore modifications to the hopping. In this limit, we impose a secondary lattice: $V_n=V\cos(2\pi\alpha n)-V$. For α irrational, the additional potential establishes an incommensurate pseudorandom model, the Harper model (for $\Omega=0$ and $V_D=0$). The potential $V_D f_n$ adds disorder where f_n is a random number satisfying $0\leqslant f_n\leqslant 1$ for each site. The confinement potential, Ωn^2, applies to optical lattice systems.

According to the Aubry-Andre conjecture [16], the Harper model exhibits a metal-insulator transition at the self-dual point $V=2$. For $V<2$, all states are extended, while for $V>2$, all states localize (the states at $V=2$ are critical). The localized states are characterized by a nonzero Lyapunov exponent (inverse localization length), $\gamma(E)$, where $u_n(E)\sim\exp(-\gamma n)$, and gaps in the energy spectra. While exceptions to the Aubry-Andre conjecture have been rigorously proven for specific values of α [17], we discuss an additional and experimentally relevant counterexample defined by $\alpha=m\pm\epsilon$, for integer m and

$$N^{-1}\ll\epsilon\ll 1, \qquad (2)$$

with N sites. In the limit $N\to\infty$, the secondary lattice defines a slowly varying, nearly periodic potential. A similar, slowly varying potential has been considered in the context of one-dimensional localization in quasiperiodic systems [18].

In the limit defined by Eq. (2), the eigenstates of Eq. (1) with $\Omega=0$ and $V_D=0$ display Anderson-like localization where we expect to find *only* extended states. To see this, consider γ defined in the limit, $N\to\infty$ [19]:

$$\gamma(E_j)=\frac{1}{N}\sum_n \ln\left|\frac{u_{n+1}}{u_n}\right|=\frac{1}{(N-1)}\sum_{j\neq l}\ln|E_j-E_l|. \qquad (3)$$

The first equality allows us to use the transfer matrix method to calculate γ for large system sizes. The solid line in the top panel of Fig. 1 plots the Lyapunov exponent vs energy for $N=10^7$ and $V=0.5$. The additional potential, V_n, shifts the lower band edge to $E=-2-2V$, while leaving the upper band edge at $E=2$. We see extended states in the center of the band, $-2<E<2-2V$, with $\gamma=0$, as expected from the Aubry-Andre conjecture. However, near the band edges, $-2-2V<E<-2$ and $2-2V<E<2$, the states localize, $\gamma>0$. The points $E=-2$ and $2-2V$ define mobility edges, which are unexpected in one dimension but found in three-dimensional models with disorder. The localization is, in this sense, Anderson-like. We find that, for $N=10^7$, the mobility edges persist for rational *and* irrational values of ϵ from 10^{-5} to 10^{-2}. We conjecture that in the limit Eq. (2) irrational numbers are approximated by rational numbers up to a number much smaller than N^{-1}. For $N\to\infty$, the spectra can contain an infinite number of infinitely small gaps, and therefore localized states, eliminating the distinction between an incommensurate and a commensurate system [12,17].

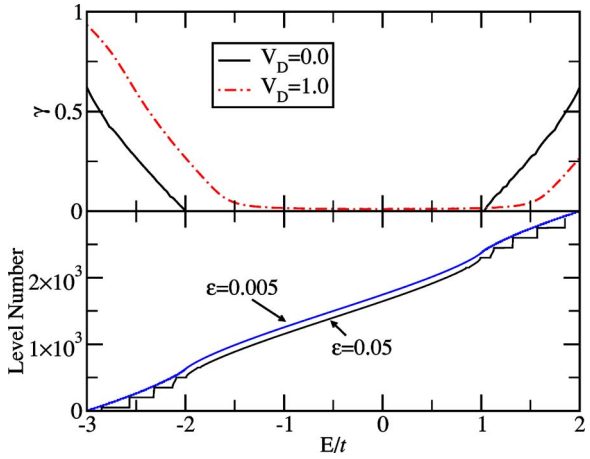

FIG. 1. (Color online) Top panel: Lyapunov exponent vs energy for the nearly periodic Harper model, Eq. (1) with $V=0.5$, $\epsilon=0.005$, and $\Omega=0$. The solid (dot-dashed) line shows the disorder-free, $V_D=0$ (disordered, $V_D=1$) case. Bottom panel: Level number vs energy of the disorder-free, nearly periodic Harper model for two characteristic values of ϵ with $V=0.5$, $N=3000$, and $\Omega=0$. The lower curve is shifted downward by 100 levels for clarity.

The unexpected insulating behavior coincides with a devil's staircase-like structure in part of the energy spectrum [12,14]. The second equality in Eq. (3) shows that a degeneracy at E_j supports nonzero $\gamma(E_j)$. The lower panel of Fig. 1 plots the level number as a function of energy as determined by the exact diagonalization of Eq. (1) for $N=3000$, $V=0.5$, $V_D=0$, and $\Omega=0$. The top line ($\epsilon=0.005$) shows smaller gaps and narrower steps than the bottom line ($\epsilon=0.05$), suggesting that the localized states develop a gapless insulator in the limits $\epsilon\to 0$ and $\epsilon\gg N^{-1}$.

We can understand the insulating states in a "semiclassical" approximation where ϵ plays the role of $\hbar$. We analyze the behavior of each regime as a function of energy. At low energies, $E<-2$, the slowly varying potential confines low-energy states near the potential minima defined by V_n. Very little tunneling between minima forces localization. Intermediate energies, $-2<E<2-2V$, see a smaller barrier between minima allowing for extended states, and therefore, the first mobility edge at $E=-2$. A second mobility edge forms at

$E=2-2V$ when states localize at the secondary lattice maxima. At first this seems counterintuitive, but can be understood in a WKB approximation based on the slowly varying nature of V_n. A similar analysis was performed for a different model in Ref. [18]. Our results show that the high-energy states, $E>2-2V$, moving energetically above the lattice, slow when passing secondary lattice maxima to force localization. We have checked that our analysis based on the WKB approximation reproduces the solid line in the upper panel of Fig. 1. As an additional check we can, in a continuum approximation [12,20], define a position variable, $n\to\tilde{x}$, and a difference operator, $u_{n+1}+u_{n-1}\to 2\cos(p)u(\tilde{x})$ (with $p\equiv i\partial/\partial\tilde{x}$), to give the semiclassical Hamiltonian: $H_{\mathrm{CL}}=-2\cos(p)+V(\tilde{x})$, with the replacement $V_n\to V(\tilde{x})$. The phase trajectories of H_{CL} produce extended and localized states (and therefore, mobility edges) in the regimes obtained in Fig. 1.

We now discuss the possibility of observing this unique type of localization. In the solid state, a necessary correction to the Harper model includes disorder where we add to V_n a potential of the form $V_D f_n$. For $V=0$ (and $\Omega=0$) this defines the one-dimensional Anderson model where we expect all states to localize for arbitrary V_D. However, for $V\neq 0$, the states (otherwise extended in the $V_D=0$ case) have a small localization length, which could allow some remnant of a mobility edge. The dot-dashed line in the upper panel of Fig. 1 plots γ for the same parameters as the solid line, but with $V_D=1.0$. We find that a finite amount of disorder obscures the position of the remnant-mobility edges, while localizing all states.

In what follows, we consider an essentially disorder-free manifestation of Eq. (1): one-dimensional, cold-atom optical lattices. The interference of appropriately detuned lasers of wavelength $\lambda=2a$ can give rise to our tight-binding lattice with a sufficiently strong lattice height V_L. To create a secondary modulating potential, V_n, consider an additional pair of lasers at angles θ and $\pi-\theta$ to the primary lattice with wavelength λ' and amplitude V_L'. The additional lasers interfere to modulate the energy of the nth site by $V_L'\cos(2\pi n\alpha)\int_{-\infty}^{\infty}|w(u)|^2\cos(2\pi u\alpha)du-V_L'$, where $\alpha=(\lambda/\lambda')\cos(\theta)$. For small angles we can retrieve, up to an overall constant, our nearly periodic Harper model with $m=\lambda/\lambda'$ an integer and $\epsilon\approx-\lambda\theta^2/2\lambda'$. For realistic parameters, $V_L=5E_R$, $V_L'=0.1E_R$, $\theta=5°$, and $\lambda=\lambda'$ (where E_R is the photon recoil energy), we find $t\approx 0.065E_R$, $V\approx 0.055E_R$, and $\epsilon\approx 0.004$ yielding the appropriate parameter regime. Furthermore, we find that, in the limit of Eq. (2), fluctuations in the relative phase do not alter the position of the mobility edge. We now include an important modification to the model, which accounts for realistic finite size effects.

A crucial addition to the Harper model in optical lattices is the parabolic confinement, Ωn^2, which leads to a finite particle number. We find that weak confinement leaves the mobility edges intact. To see this, consider the local Lyapunov exponent, $\gamma^L(E_j)=(2N_{\mathrm{CL}}+1)^{-1}\Sigma_{n=-N_{\mathrm{CL}}}^{N_{\mathrm{CL}}}\ln|u_{n+1}/u_n|$, where the semiclassical limits of the parabolic trap define the number of states participating in transport, $2N_{\mathrm{CL}}+1$. The classical turning points give $N_{\mathrm{CL}}=2|x_{\mathrm{CL}}(E)|$ and Eq. (2) becomes $(2N_{\mathrm{CL}})^{-1}\ll\epsilon\ll 1$. To determine x_{CL}, we set $p=0$ in

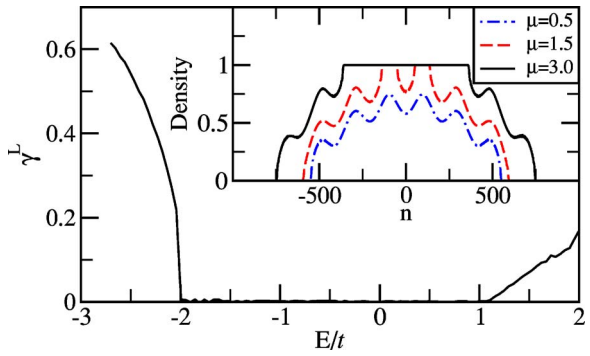

FIG. 2. (Color online) The local Lyapunov exponent vs energy for the same parameters as the upper panel of Fig. 1, but with an additional parabolic confinement $\Omega=10^{-5}$ and no disorder, $V_D=0$. The inset shows the normalized density of the same system as a function of the lattice number for several different chemical potentials.

H_{CL} with $V(\tilde{x})=V\cos(2\pi\alpha\tilde{x})-V+\Omega\tilde{x}^2$. For $\Omega\sim 10^{-5}$, we find $2N_{\mathrm{CL}}\sim 10^3$. In the limit $\Omega\to 0$, we retrieve the usual Lyapunov exponent, $\gamma^L\to\gamma$. Figure 2 plots the local Lyapunov exponent as a function of energy for $N=10^7$, $\epsilon=0.005$, $V=0.5$, $V_D=0$, and $\Omega=10^{-5}$. The mobility edges remain even with a reduced number of states comprising the system. The inset shows the normalized density profile as a function of the site number for three different chemical potentials, μ. At zero temperature, we include states with $E\leqslant\mu$. For $\mu=0.5$ (dashed-dotted line), we find extended states with some modulation due to V_n. For $\mu=1.5$ (dashed line), we have crossed the mobility edge and the density pins to unity at some lattice sites. The formation of a mesoscopic version of the Anderson-like gapless insulator fixes the density. For $\mu=3.0$, we enter the band-insulator regime, which fixes a large fraction of the states at integer density.

Dipole oscillations in harmonically confined atomic gases serve as a direct probe of localization [7,8]. A small shift in the center-of-mass results in harmonic oscillations in the absence of an external lattice. The presence of one or more weak lattices allows for weakly localized states, which can suppress oscillations and lead to an effective underdamping of the center-of-mass motion. The addition of strongly localized states can, in the absence of dissipation, eventually pin the center-of-mass to effectively overdamp the center-of-mass oscillations. Strong experimental and theoretical evidence supports the possibility that band localization has indeed been observed in fermionic, one-dimensional optical lattices [7]. Similar evidence also suggests such behavior for strongly interacting bosons [8].

We now study the onset of the gapless Anderson-like insulator and its effect on center-of-mass oscillations. Consider the center-of-mass to be displaced Δ lattice sites at some initial time $T=0$. For extended states, the center-of-mass position, $\bar{X}(T)$, averages to zero for long times, while localized states should pin the center-of-mass position, $\bar{X}\sim\Delta$. The center-of-mass position can, for some parameters, demonstrate complex, damping-like behavior as function of time making a damping constant ill defined. To extract a simple quantity to be compared with experiment, we calculate the long-time average of the center-of-mass position, $\langle\bar{X}\rangle_\infty$, as a

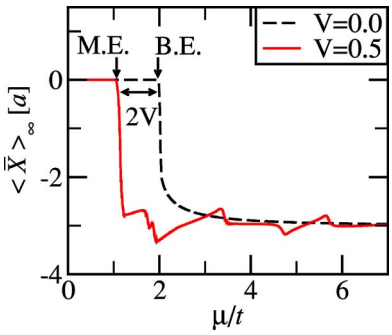

FIG. 3. (Color online) The long-time average of the center-of-mass position as a function of the chemical potential after an initial displacement of three sites, $\Delta=-3$, with no disorder, $V_D=0$. The dashed line is calculated from the bare tight-binding model with no secondary lattice, $V=0$, and $\Omega=10^{-5}$. B.E. labels the upper band edge. The solid line is calculated for the same parameters, but with $V=0.5$. The early onset of the gapless insulator occurs at the upper mobility edge, labeled M.E., while the upper band edge remains at $\mu=2$.

function of chemical potential by diagonalizing Eq. (1) with a parabolic potential, $\Omega=10^{-5}$, for $N=3000$, $\Delta=-3$, and $V_D=0$. As an intermediate step we require degenerate eigenstates (localized at the edges) to simultaneously diagonalize the parity operator, since our system possesses reflection symmetry about the origin. The dashed line in Fig. 3 plots $\langle\bar{X}\rangle_\infty$ as a function of the chemical potential in the absence of a secondary lattice, $V=0$. For $\mu<2$, the extended states perform several oscillations about the trap center, but over long times average to zero displacement. Above the band edge (labeled B.E.), for $\mu>2$, localized states near the edge pin the center of mass near Δ. For $\mu\gtrsim 3$ the system never leaves its initial position.

A second, weaker lattice causes a mobility edge to form energetically below the band edge. The solid line in Fig. 3 plots the same as the dashed line but with a second lattice, V_n, with $V=0.5$ for the chemical potentials near the upper mobility edge (labeled M.E.). $\langle\bar{X}\rangle_\infty$ remains zero where we expect extended states but pins near Δ for $\mu>2-2V$. The mesoscopic version of the gapless insulator results in the early onset of pinning in the regime $2-2V<\mu<2$ and Eq. (2). Furthermore, the localized states with the additional lattice, $V=0.5$, also display weak periodicity in $\langle\bar{X}\rangle_\infty$ as a function of μ. These oscillations correspond to the chemical potential passing through peaks and valleys in the corrugated confinement potential.

Fluctuations in the lattice depth can soften the otherwise sharp mobility edge. The quantity of interest, $2V/t$, can fluctuate wildly with only moderate changes in V_L at extremely large lattice depths. To see this, consider an approximate expression in terms of the hopping extracted from an analysis of the related Mathieu problem, $V/t\approx(\sqrt{\pi}V/4)(V_L/E_R)^{-3/4}\exp(2\sqrt{V_L/E_R})$. A relative error in V and V_L, R_V and R_{V_L}, respectively, propagates to a relative error in $2V/t$: $[R_V^2+R_{V_L}^2(3/4-\sqrt{V_L/E_R})^2]^{1/2}$. We have checked that this formula is quantitatively accurate for $V_L\gtrsim 5E_R$ by comparing with the error derived numerically from the exact tunneling. We find that for $R_V=R_{V_L}=5\%$, the relative error in $2V/t$ remains below 20% for $V_L<20E_R$.

We note that the additional time dependence in the model discussed possesses other applications. We take H_{CL} as a good approximation to the nearly periodic Harper model in the limit Eq. (2). In the presence of a pulsed secondary lattice, $V\propto\Sigma_j\delta(T-jT_0)$, where for integer j the secondary lattice oscillates with period T_0, we simulate the kicked Harper model via H_{CL}. The kicked Harper model exhibits chaotic behavior with the "classical" to quantum crossover controlled by ϵ.

We have explicitly demonstrated the existence of a mobility edge (and the associated, unusual metal-insulator transition in a deterministic disorder-free environment) in suitably designed aperiodic cold-atom optical lattice systems. The deterministic aperiodic background potential in these optical lattices leads to exotic and nontrivial energy eigenstates dependent on the relationship between irrational numbers and their rational approximations. The ensuing quantum localization occurs in the absence of disorder, and therefore distinguishes itself from Anderson localization which, in the solid state, masks the presence of mobility edges formed from quasiperiodic potentials in one dimension.

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